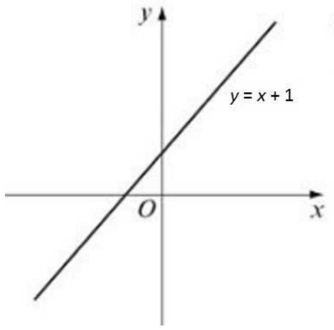
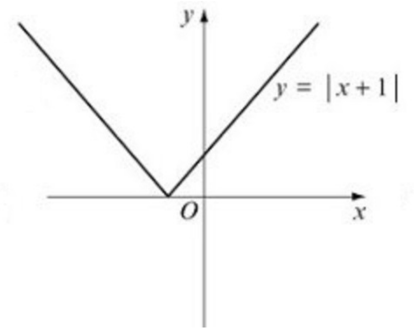
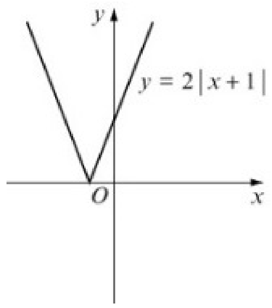
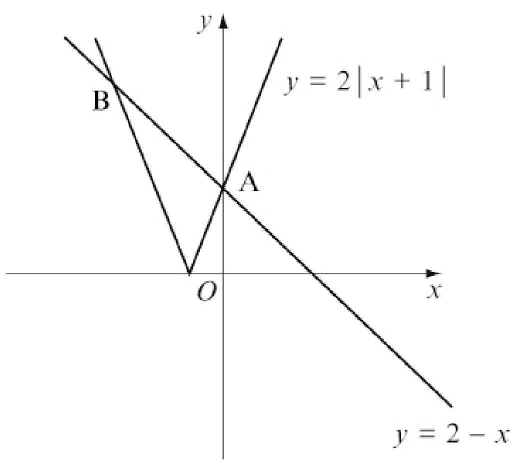
Chapter review

1 a









1 b Intersection point A:

$2(x+1) = 2 - x$

$2x + 2 = 2 - x$

$3x = 0$

$x = 0$

Intersection point B is on the reflected part of the modulus graph.

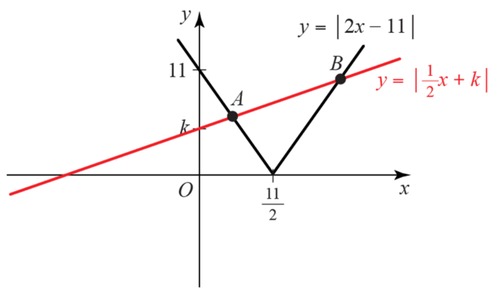
$-2(x+1) = 2 - x$

$-2x - 2 = 2 - x$

$-x = 4$

$x = -4$

2



Minimum value of $y = |2x - 11|$ is

$y = 0$ at $x = \dfrac{11}{2}$

For two distinct solutions to

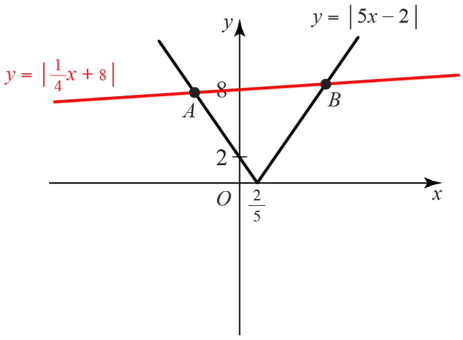
$|2x - 11| = \dfrac{1}{2}x - k$, we must have

$\dfrac{1}{2}x - k > 0$ at $x = \dfrac{11}{2}$

$\dfrac{1}{2} \times \dfrac{11}{2} + k > 0$

$k > -\dfrac{11}{4}$

3



At A:

$$-(5x-2) = -\frac{1}{4}x + 8$$
$$-20x + 8 = -x + 32$$
$$-19x = 24$$
$$x = -\frac{24}{19}$$

At B:

$$5x - 2 = -\frac{1}{4}x + 8$$
$$20x - 8 = -x + 32$$
$$21x = 40$$
$$x = \frac{40}{21}$$

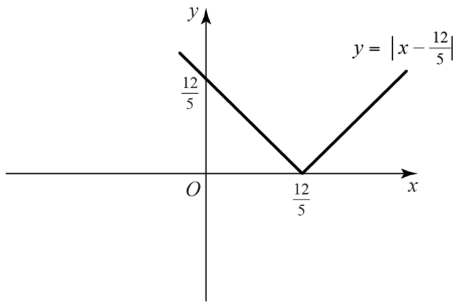
So the solution are

$$x = -\frac{24}{19} \text{ and } x = \frac{40}{21}$$

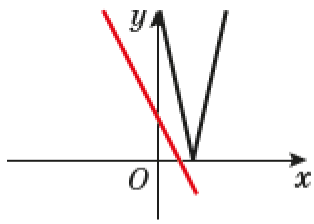
4 a $y = |12 - 5x| = 5\left|-\left(x - \frac{12}{5}\right)\right|$

Start with $y = |x|$

$y = \left|x - \frac{12}{5}\right|$ is a horizontal translation of $+\frac{12}{5}$



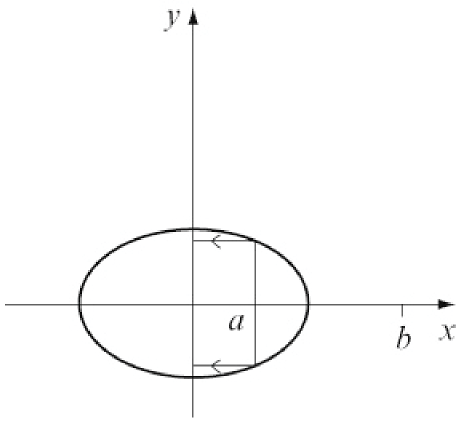
$y = 5\left|x - \frac{12}{5}\right|$ is a vertical stretch, scale factor 5



b The graphs do not intersect, so there are no solutions.

5 a i One-to-many.

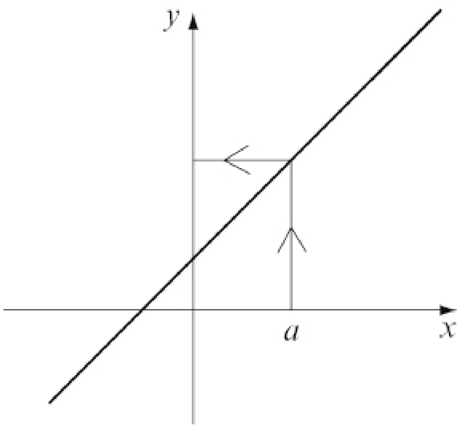
ii Not a function.



x value a gets mapped to two values of y.
x value b gets mapped to no values.

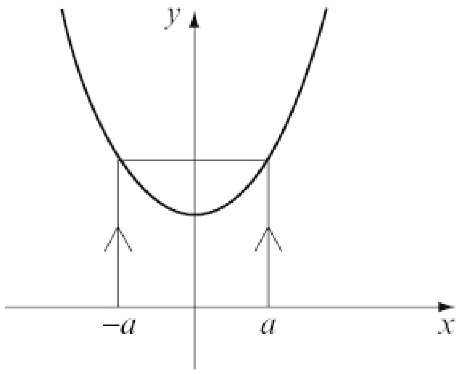
b i One-to-one.

ii Is a function.



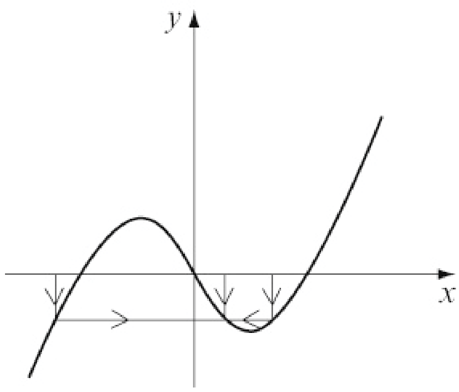
5 c i Many-to-one.

ii Is a function.



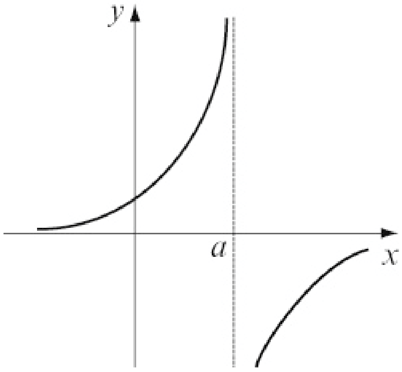
d i Many-to-one.

ii Is a function.



5 e i One-to-one.

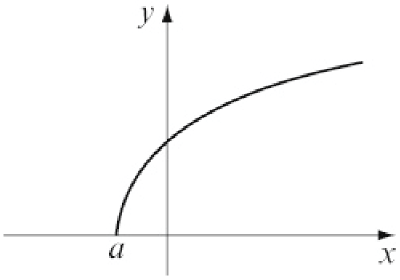
ii Not a function.



x value a doesn't get mapped to any value of y. It could be redefined as a function if the domain is said to exclude point a.

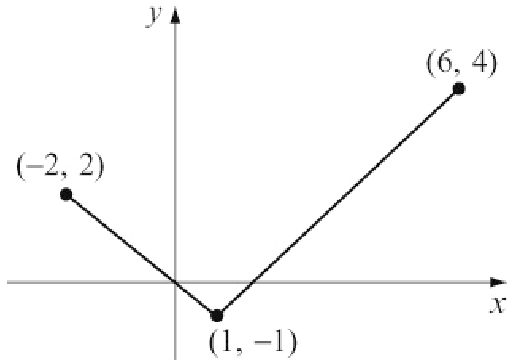
f i One-to-one.

ii Not a function for this domain.



x values less than a don't get mapped anywhere. Again, we could define the domain to be $x \leqslant a$ and then it would be a function.

6 a



For $x \leqslant 1$, $f(x) = -x$

This is a straight line of gradient -1.

At point $x = 1$, its y-coordinate is -1.

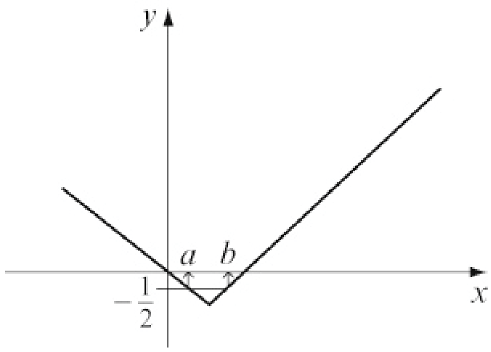
For $x > 1$, $f(x) = x - 2$

This is a straight line of gradient $+1$.

At point $x = 1$, its y-coordinate is also -1.

Hence, the graph is said to be continuous.

b There are two values x in the range $-2 \leqslant x \leqslant 6$ for which $f(x) = -\frac{1}{2}$



Point a is where

$$-x = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

Point b is where

$$x - 2 = -\frac{1}{2} \Rightarrow x = 1\frac{1}{2}$$

Hence, the values of x for which $f(x) = -\frac{1}{2}$ are $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$

7 a $\text{pq}(x) = \text{p}(2x + 1)$
$= (2x + 1)^2 + 3(2x + 1) - 4$
$= 4x^2 + 4x + 1 + 6x + 3 - 4$
$= 4x^2 + 10x$

b $\text{qq}(x) = \text{q}(2x + 1)$
$= 2(2x + 1) + 1$
$= 4x + 3$

$\text{pq}(x) = \text{qq}(x)$ gives
$4x^2 + 10x = 4x + 3$
$4x^2 + 6x - 3 = 0$

Using the formula:

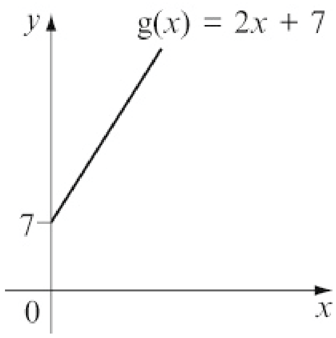
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 4 \times (-3)}}{2 \times 4}$$

$$x = \frac{-6 \pm \sqrt{84}}{8}$$

$$x = \frac{-6 \pm 2\sqrt{21}}{8}$$

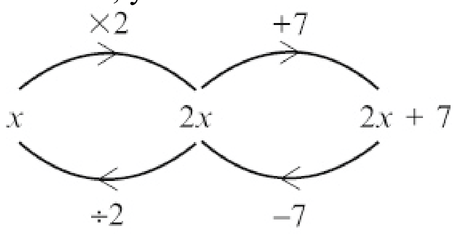
$$x = \frac{-3 \pm \sqrt{21}}{4}$$

8 a $y = 2x + 7$ is a straight line with gradient 2 and y-intercept 7



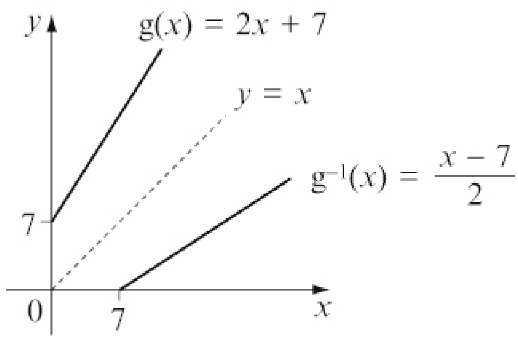
For $x \geqslant 0$, the range is $\text{g}(x) \geqslant 7$

8 b The range is $g^{-1}(x) \geqslant 0$.
To find the equation of the inverse function, you can use a flow chart.



$g^{-1}(x) = \dfrac{x-7}{2}$ and has domain $x \geqslant 7$

c



$\text{g}^{-1}(x)$ is the reflection of $\text{g}(x)$ in the line $y = x$.

9 a To find $\text{f}^{-1}(x)$, you can change the subject of the formula.

$$\text{Let } y = \frac{2x+3}{x-1}$$

$$y(x-1) = 2x+3$$

$$yx - y = 2x + 3$$

$$yx - 2x = y + 3$$

$$x(y-2) = y+3$$

$$x = \frac{y+3}{y-2}$$

$$\text{Therefore } \text{f}^{-1}(x) = \frac{x+3}{x-2}$$

9 b i Domain $\text{f}(x) =$ Range $\text{f}^{-1}(x)$

$\therefore$ Range $\text{f}^{-1}(x) = \{y \in \mathbb{R}, y > 1\}$

ii Range $\text{f}(x) =$ Domain $\text{f}^{-1}(x)$

Now range of $\text{f}(x)$ is

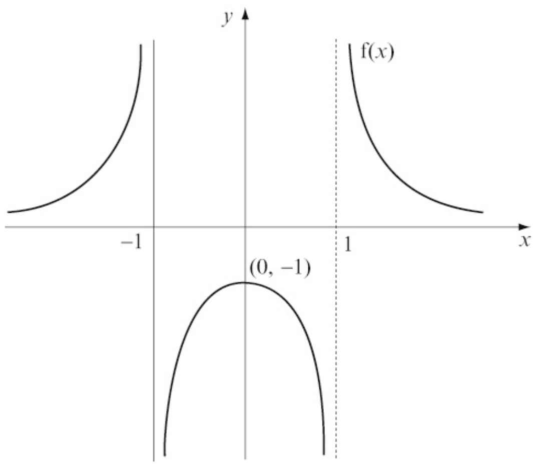
$\{\text{f}(x) \in \mathbb{R}, \text{f}(x) > 2\}$

$\therefore$ Domain $\text{f}^{-1}(x) = \{x \in \mathbb{R}, x > 2\}$

10 a
$$\text{f}(x) = \frac{x}{x^2-1} - \frac{1}{x+1}$$
$$= \frac{x}{(x+1)(x-1)} - \frac{1}{(x+1)}$$
$$= \frac{x}{(x+1)(x-1)} - \frac{x-1}{(x+1)(x-1)}$$
$$= \frac{x-(x-1)}{(x+1)(x-1)}$$
$$= \frac{1}{(x+1)(x-1)}$$

b Consider the graph of

$y = \dfrac{1}{(x-1)(x+1)}$ for $x \in \mathbb{R}$:



For $x > 1$, $\text{f}(x) > 0$

10 c
$$\text{gf}(x) = \text{g}\left(\frac{1}{(x-1)(x+1)}\right)$$
$$= \frac{2}{\left(\frac{1}{(x-1)(x+1)}\right)}$$
$$= 2 \times \frac{(x-1)(x+1)}{1}$$
$$= 2(x-1)(x+1)$$

$$\text{gf}(x) = 70 \Rightarrow 2(x-1)(x+1) = 70$$
$$(x-1)(x+1) = 35$$
$$x^2 - 1 = 35$$
$$x^2 = 36$$
$$x = \pm 6$$

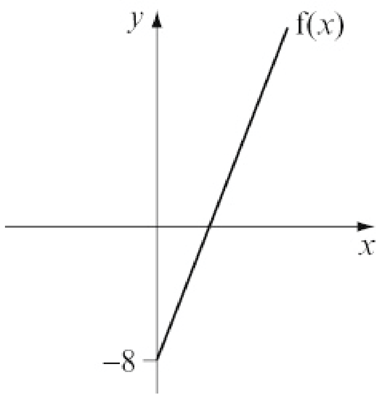
11 a
$$\text{f}(7) = 4(7-2)$$
$$= 4 \times 5$$
$$= 20$$
$$\text{g}(3) = 3^3 + 1$$
$$= 27 + 1$$
$$= 28$$

$$\text{h}(-2) = 3^{-2}$$
$$= \frac{1}{3^2}$$
$$= \frac{1}{9}$$

11 b $f(x) = 4(x-2) = 4x-8$

This is a straight line with gradient 4 and intercept −8.

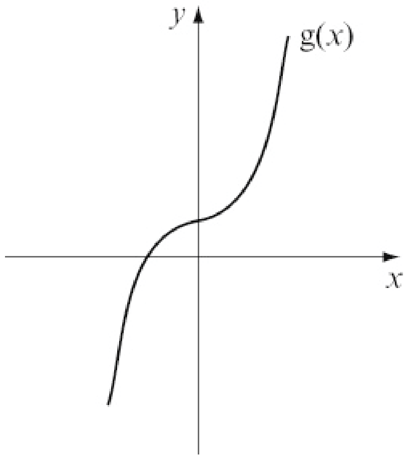
The domain tells us that $x \geqslant 0$, so the graph of $y = f(x)$ is:



The range of $f(x)$ is

$f(x) \in \mathbb{R}, f(x) \geqslant -8$

$g(x) = x^3 + 1$



The range of $g(x)$ is $g(x) \in \mathbb{R}$

c Let $y = x^3 + 1$

(change the subject of the formula)

$y - 1 = x^3$

$\sqrt[3]{y-1} = x$

Hence $g^{-1}(x) = \sqrt[3]{x-1} \quad \{x \in \mathbb{R}\}$

d $fg(x) = f(x^3+1)$

$= 4(x^3+1-2)$

$= 4(x^3-1)$

11 e First find $gh(x)$:

$gh(x) = g(3^x)$

$= (3^x)^3 + 1$

$= 3^{3x} + 1$

$gh(a) = 244$

$3^{3a} + 1 = 244$

$3^{3a} = 243$

$3^{3a} = 3^5$

$3a = 5$

$a = \frac{5}{3}$

12 a f^{-1} exists when f is one-to-one.

Now $f(x) = x^2 + 6x - 4$

Completing the square:

$f(x) = (x+3)^2 - 13$

The minimum value is

$f(x) = -13$ when $x + 3 = 0$

$\Rightarrow x = -3$

Hence, f is one-to-one when $x > -3$

So least value of a is $a = -3$

b Let $y = f(x)$

$y = x^2 + 6x - 4$

$y = (x+3)^2 - 13$

$y + 13 = (x+3)^2$

$x + 3 = \sqrt{y+13}$

$x = \sqrt{y+13} - 3$

So $f^{-1} : x \mapsto \sqrt{x+13} - 3$

For $a = 0$, Range $f(x)$ is $y > -4$

So Domain $f^{-1}(x)$ is $x > -4$

13 a $f: x \mapsto 4x-1$

Let $y = 4x - 1$ and change the subject of the formula.

$\Rightarrow y+1 = 4x$

$\Rightarrow x = \frac{y+1}{4}$

Hence $f^{-1}: x \mapsto \frac{x+1}{4}, \ x \in \mathbb{R}$

b $gf(x) = g(4x-1)$

$= \frac{3}{2(4x-1)-1}$

$= \frac{3}{8x-3}$

Hence $gf: x \mapsto \frac{3}{8x-3}$

$gf(x)$ is undefined when $8x-3=0$

That is, at $x = \frac{3}{8}$

$\therefore$ Domain $gf(x) = \left\{x \in \mathbb{R}, x \neq \frac{3}{8}\right\}$

c If $2f(x) = g(x)$

$2 \times (4x-1) = \frac{3}{2x-1}$

$8x - 2 = \frac{3}{2x-1}$

$(8x-2)(2x-1) = 3$

$16x^2 - 12x + 2 = 3$

$16x^2 - 12x - 1 = 0$

Use $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

with $a = 16$, $b = -12$ and $c = -1$.

Then $x = \frac{12 \pm \sqrt{144+64}}{32}$

$= \frac{12 \pm \sqrt{208}}{32}$

$= 0.826, -0.076$

Values of x are -0.076 and 0.826

14 a Let $y = \frac{x}{x-2}$

$y(x-2) = x$

$yx - 2y = x$ (rearrange)

$yx - x = 2y$

$x(y-1) = 2y$

$x = \frac{2y}{y-1}$

$f^{-1}(x) = \frac{2x}{x-1}, \ x \neq 1$

b The range of $f^{-1}(x)$ is the domain of $f(x)$:

$\{f^{-1}(x) \in \mathbb{R}, \ f^{-1}(x) \neq 2\}$

c $gf(1.5) = g\left(\frac{1.5}{1.5-2}\right)$

$= g\left(\frac{1.5}{-0.5}\right)$

$= g(-3)$

$= \frac{3}{-3}$

$= -1$

d If $g(x) = f(x) + 4$

$\frac{3}{x} = \frac{x}{x-2} + 4$

$3(x-2) = x^2 + 4x(x-2)$

$3x - 6 = x^2 + 4x^2 - 8x$

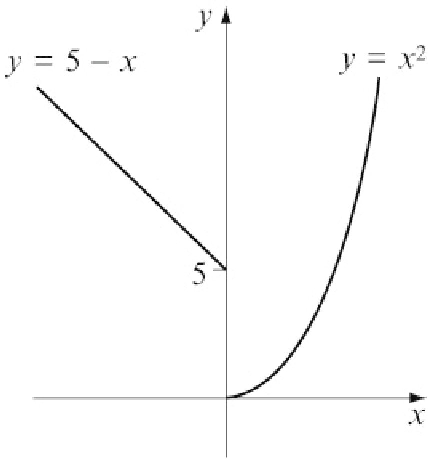
$0 = 5x^2 - 11x + 6$

$0 = (5x-6)(x-1)$

$\Rightarrow x = \frac{6}{5}, 1$

15 $y = 5 - x$ is a straight line with gradient -1 passing through 5 on the y axis.

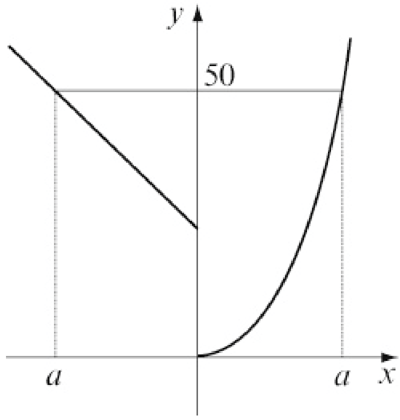
$y = x^2$ is a $\cup$-shaped quadratic passing through $(0, 0)$



a $n(-3) = 5 - (-3)$
$= 5 + 3$
$= 8$

$n(3) = 3^2$
$= 9$

15 b From the diagram, you can see there are two values of x for which $n(x) = 50$



The negative value of x is where

$5 - x = 50$

$x = 5 - 50$

$x = -45$

The positive value of x is where

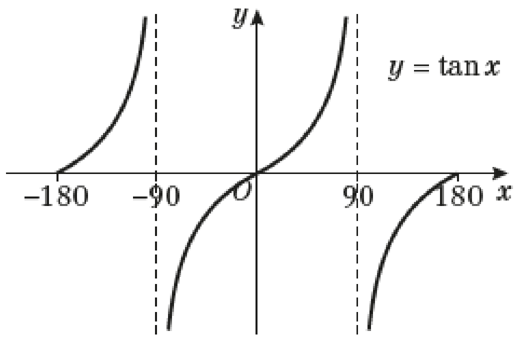
$x^2 = 50$

$x = \sqrt{50}$

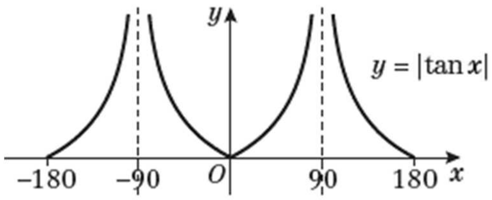
$x = 5\sqrt{2}$

The values of x such that $n(x) = 50$ are -45 and $+5\sqrt{2}$

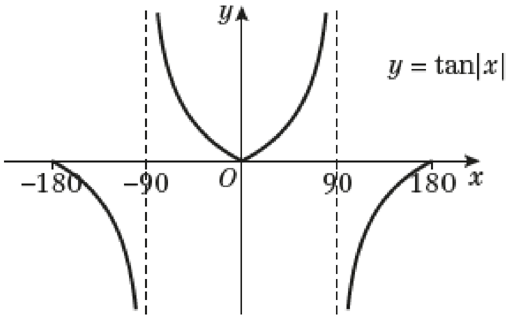
16 a



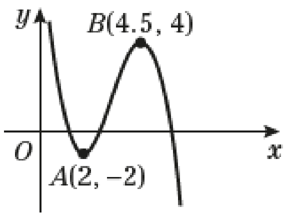
16 b $y = |\tan(x)|$ reflects the negative parts of $\tan x$ in the x axis.



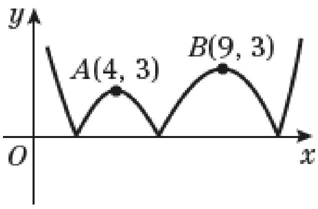
c $y = \tan(|x|)$ reflects $\tan x$ in the y-axis.



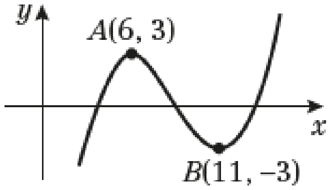
17 a



b



c

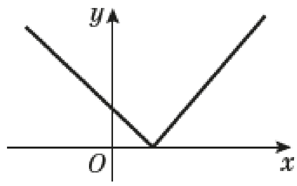


18 a $g(x) \geqslant 0$

b $gf(x) = g(4 - x)$
$= 3(4 - x)^2$
$= 3x^2 - 24x + 48$

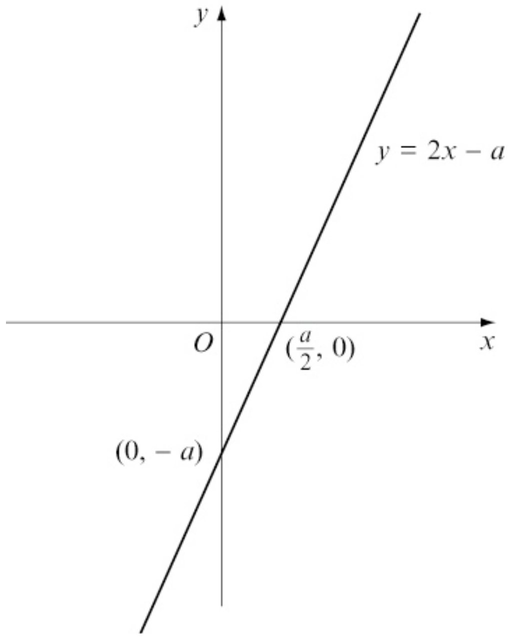
$gf(x) = 48$
$3x^2 - 24x + 48 = 48$
$3x^2 - 24x = 0$
$3x(x - 8) = 0$
$x = 0$ or $x = 8$

c



$|f(x)| = 2$ when $|4 - x| = 2$, so
$4 - x = 2 \Rightarrow x = 2$
or $-(4 - x) = 2 \Rightarrow x = 6$

19 a

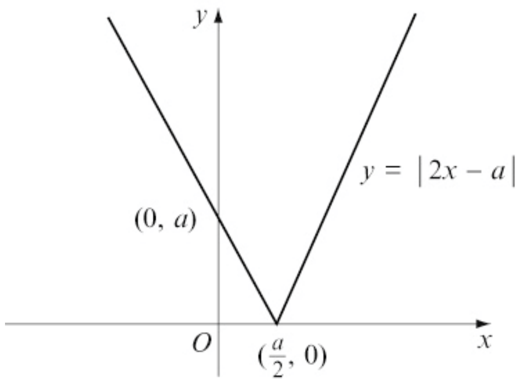


For $y = |2x - a|$:

When $x = 0$, $y = |-a| = a$ $\quad (0, a)$

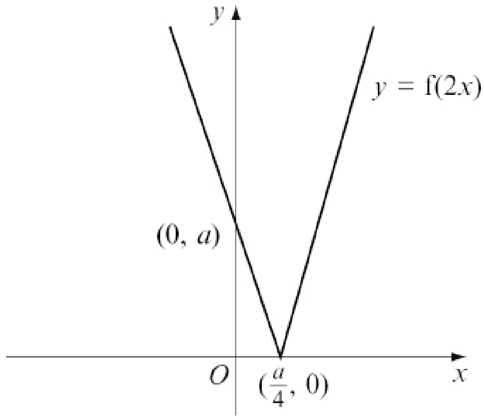
When $y = 0$, $2x - a = 0$

$\Rightarrow x = \frac{a}{2}$ $\quad \left(\frac{a}{2}, 0\right)$



19 b $y = \mathrm{f}(2x)$

Horizontal stretch, scale factor $\frac{1}{2}$



c $|2x - a| = \frac{1}{2}x$

Either $(2x - a) = \frac{1}{2}x$

$\Rightarrow a = \frac{3}{2}x$

Given that $x = 4$,

$a = \frac{3 \times 4}{2} = 6$

Or

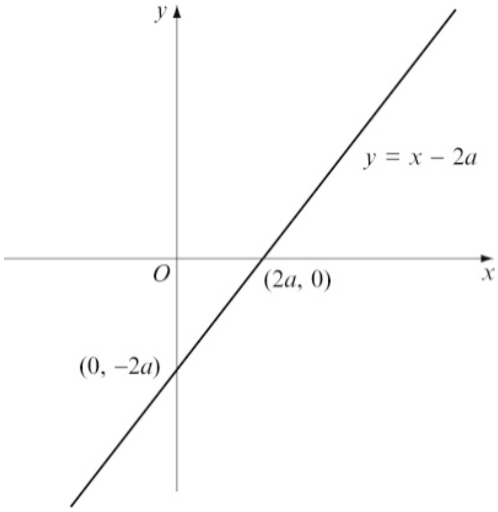
$-(2x - a) = \frac{1}{2}x$

$\Rightarrow a = \frac{5}{2}x$

Given that $x = 4$,

$a = \frac{5 \times 4}{2} = 10$

20 a

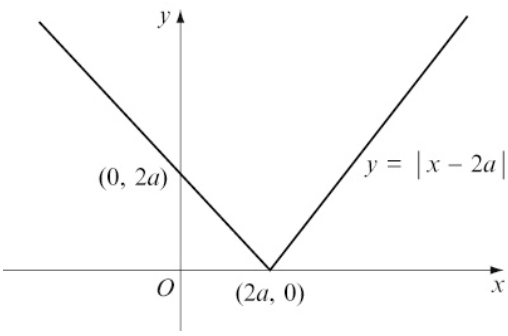


For $y = |x - 2a|$:

When $x = 0$, $y = |-2a| = 2a$ $(0, 2a)$

When $y = 0$, $x - 2a = 0$

$\Rightarrow x = 2a$ $(2a, 0)$



20 b $|x - 2a| = \frac{1}{3}x$

Either $(x - 2a) = \frac{1}{3}x$

$\Rightarrow x - \frac{1}{3}x = 2a$

$\Rightarrow \frac{2}{3}x = 2a$

$\Rightarrow x = 3a$

or $-(x - 2a) = \frac{1}{3}x$

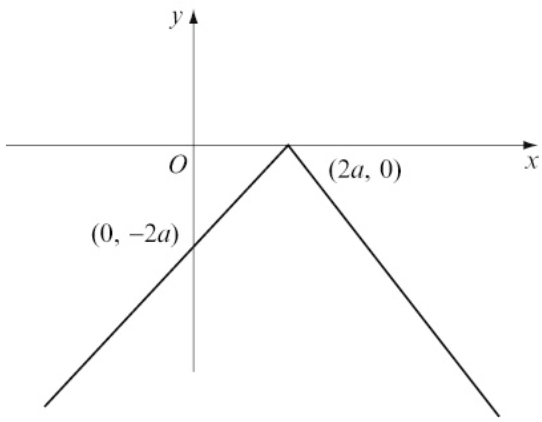
$\Rightarrow -x + 2a = \frac{1}{3}x$

$\Rightarrow \frac{4}{3}x = 2a$

$\Rightarrow x = \frac{3}{2}a$

20 c $y = -|x - 2a|$

Reflect $y = |x - 2a|$ in the x-axis



$y = a - |x - 2a|$ Vertical translation by $+a$

For $y = a - |x - 2a|$:

When $x = 0$,

$y = a - |-2a|$

$= a - 2a$

$= -a$ $(0, -a)$

When $y = 0$,

$a - |x - 2a| = 0$

$|x - 2a| = a$

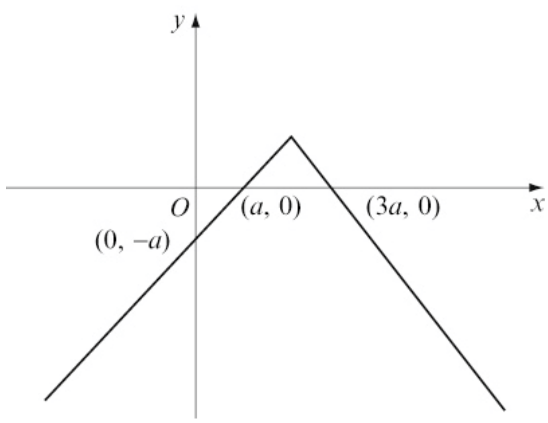
Either $x - 2a = a$

$\Rightarrow x = 3a$ $(3a, 0)$

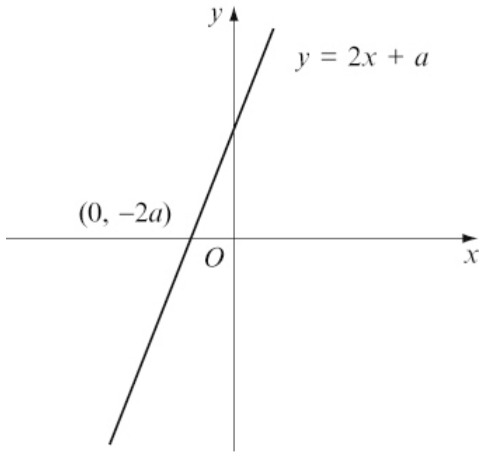
or $-(x - 2a) = a$

$\Rightarrow -x + 2a = a$

$\Rightarrow x = a$ $(a, 0)$



21 a & b

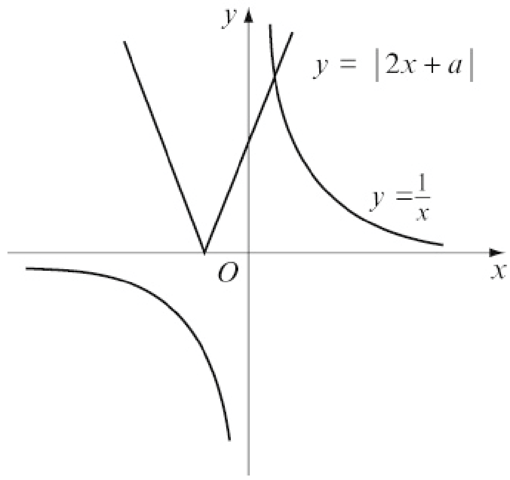


For $y = |2x + a|$:

When $x = 0$, $y = |a| = a$ $(0, a)$

When $y = 0$, $2x + a = 0$

$\Rightarrow x = -\frac{a}{2}$ $\left(-\frac{a}{2}, 0\right)$



c Intersection of graphs in **b** gives solutions to the equation:

$|2x + a| = \frac{1}{x}$

$x|x + a| = 1$

$x|2x + a| - 1 = 0$

The graphs intersect once only, so $x|2x + a| - 1 = 0$ has only one solution.

21 d The intersection point is on the non-reflected part of the modulus graph, so here $|2x-a| = 2x-a$

$x(2x+a)-1=0$

$2x^2+ax-1=0$

$$x=\frac{-a\pm\sqrt{a^2+8}}{4}$$

As shown on the graph, x is positive at intersection,

so $x=\dfrac{-a+\sqrt{a^2+8}}{4}$

22 a $f(x)=x^2-7x+5\ln x+8$

$f'(x)=2x-7+\dfrac{5}{x}$

At stationary points, $f'(x)=0$:

$2x-7+\dfrac{5}{x}=0$

$2x^2-7x+5=0$

$(2x-5)(x-1)=0$

$x=\dfrac{5}{2},\ x=1$

Point A: $x=1$,

$f(x)=1-7+5\ln 1+8$

$=2$

A is $(1,\ 2)$

Point B: $x=\dfrac{5}{2}$,

$f(x)=\dfrac{25}{4}-\dfrac{35}{2}+5\ln\dfrac{5}{2}+8$

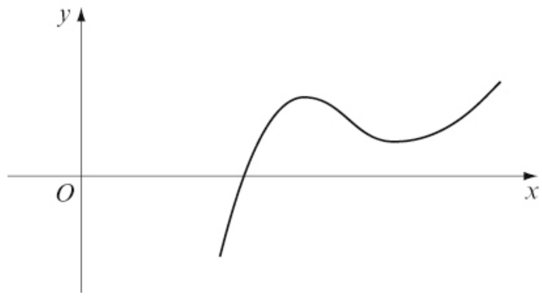
$=5\ln\dfrac{5}{2}-\dfrac{13}{4}$

B is $\left(\dfrac{5}{2}, 5\ln\dfrac{5}{2}-\dfrac{13}{4}\right)$

22 b $y=f(x-2)$

Horizontal translation of $+2$.

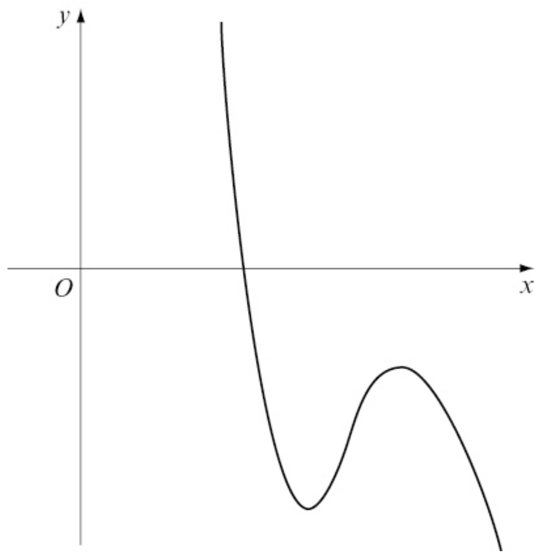
Graph looks like:



$y=-3f(x-2)$

Reflection in the x-axis, and vertical stretch, scale factor 3.

Graph looks like:



22 c Using the transformations,
point (X, Y)
becomes $(X+2, -3Y)$

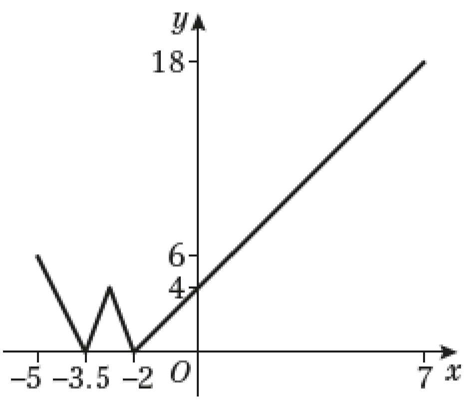
$(1,2) \to (3,-6)$
Minimum

$\left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right) \to$
$\left(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2}\right)$
Maximum

23 a The range of f(x) is $-2 \leqslant f(x) \leqslant 18$

b ff(−3) = f(−2)
Using f(x) = 2x + 4
f(−2) = 2 × (−2) + 4 = 0

c



23 d Look at each section of $f(x)$ separately.

$-5 \leqslant x \leqslant -3$:

$$\text{Gradient} = \frac{-2-6}{-3-(-5)} = -4$$

$$\therefore f(x) - (-2) = -4(x-(-3)) \Rightarrow f(x) = -4x - 14$$

So in this region, $f(x) = 2$ when $x = -4$

$\therefore fg(x) = 2$ has a corresponding solution if

$$g(x) = -4 \Rightarrow g(x) + 4 = x^2 - 7x + 14 = 0$$

Discriminant $(-7)^2 - 4(1)(14) = -7 < 0$

So no solution

$-3 \leqslant x \leqslant 7$: $\text{Gradient} = \frac{18-(-2)}{7-(-3)} = 2$

$$\therefore f(x) - (-2) = 2(x-(-3)) \Rightarrow f(x) = 2x + 4$$

So in this region, $f(x) = 2$ when $x = -1$

$\therefore fg(x) = 2$ has a corresponding solution if

$$g(x) = -1 \Rightarrow g(x) + 1 = x^2 - 7x + 11 = 0$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)} = \frac{7 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{7+\sqrt{5}}{2} \text{ or } x = \frac{7-\sqrt{5}}{2}$$

24 a The range of p(x) is p(x) $\leqslant 10$

b p(x) is many-to-one, therefore the inverse is one-to-many, which is not a function.

c At first point of intersection:
$2(x + 4) + 10 = -4$
$2x + 18 = -4$
$x = -11$
At the other point of intersection:
$-2(x + 4) + 10 = -4$
$-2x + 2 = -4$
$x = 3$
$-11 < x < 3$

24 d For no solutions, $p(x) > 10$ at $x = -4$

So $-\frac{1}{2}x + k > 10$ at $x = -4$

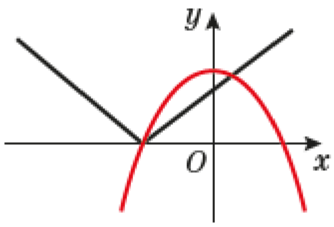
$-\frac{1}{2}(-4) + k > 10$

$2 + k > 10$

$k > 8$

Challenge

a



b $y = (a + x)(a - x)$
When $y = 0$, $x = -a$ or $x = a$
When $x = 0$, $y = a^2$
$(-a, 0)$, $(a, 0)$, $(0, a^2)$

c When $x = 4$, $y = a^2 - x^2$
$= a^2 - 16$
and $y = x + a$
$= 4 + a$
$a^2 - 16 = 4 + a$
$a^2 - a - 20 = 0$
$(a - 5)(a + 4) = 0$
As $a > 1$, $a = 5$